Journal of Mathematical Analysis and Applications **238**, 216–233 (1999)
Article ID jmaa.1999.6527, available online at http://www.idealibrary.com on IDEAL®

# Minimal Period Estimates of Periodic Solutions for Superquadratic Hamiltonian Systems

Guihua Fei, Soon-Kyu Kim, and Tixiang Wang



*Submitted by Andrea Donato*

Received November 19, 1998

In this paper, we study the minimal period problem for the first-order Hamiltonian systems which may not be strictly convex. By using variational methods and the iteration formula of the Maslov-type index theory, we obtain estimates of the minimal period of the corresponding nonconstant periodic solutions. © 1999 Academic Press

*Key Words:* minimal period problem; Hamiltonian systems; iteration formula; Maslov-type index; Galerkin approximation procedure.

## 1. INTRODUCTION AND MAIN RESULTS

In this paper, we study the minimal period problem of the Hamiltonian systems

$$\dot{x} = JH'(x), \quad \forall x \in R^{2N}, \tag{HS}$$

where $\dot{x} = dx/dt$, $H'$ denotes the gradient with respect to the variable $x$, and $J$ is the standard symplectic matrix

$$J = \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix},$$

where $I_N$ is the identity matrix on $R^N$ and $N$ is a positive integer. In [24], Rabinowitz conjectured that for any given $T > 0$ the system (HS) possesses a nonconstant periodic solution with minimal period $T$ if $H \in C^2(R^{2N}, R)$ and satisfies the following conditions:

(H1) $H(x) \geq 0$ for all $x \in R^{2N}$.

(H2) $H(x) = o(|x|^2)$ as $x \to 0$.



0022-247X/99 $30.00

(H3) There are constants $\mu > 2$ and $r_0 > 0$ such that

$$0 < \mu H(x) \leq x \cdot H'(x) \qquad \text{for } |x| \geq r_0.$$

Since then, there have been many papers on this minimal period problem (cf. [2, 3, 6–10, 12–15, 21–23] and the references therein). If $H$ is strictly convex, i.e., $H''(x)$ is positive definite except $x = 0$, Ekeland and Hofer (cf. [9]) proved that Rabinowitz's conjecture is true.

In this paper, we study the minimal period problem for the Hamiltonian systems (HS) which may not be strictly convex. We obtain estimates on the minimal period of the corresponding nonconstant periodic solutions of (HS). The first result is the following theorem.

THEOREM 1.1. *Suppose that $H \in C^2(R^{2N}, R)$ satisfies* (H1)–(H3) *and*

(H4) *$H''(x)$ is semipositive definite for all $x \in R^{2N}$.*

*Then for every $T > 0$, the system* (HS) *possesses a nonconstant $T$-periodic solution with the minimal period not smaller than $T/(2N)$.*

Next we consider the minimal period problem for $H(x) = \frac{1}{2}h_0 x \cdot x + \hat{H}(x)$, where $h_0 \in \mathscr{L}_s(R^{2N})$, the set of all symmetric real $2N \times 2N$ matrices. This is motivated by [12] and [24], where they considered the case in which $h_0$ is a positive definite matrix.

For any $h \in \mathscr{L}_s(R^{2N})$ and any $T > 0$, let

$$D_m(h) = \begin{pmatrix} -\dfrac{T}{2\pi}h & -mJ \\ mJ & -\dfrac{T}{2\pi}h \end{pmatrix} \qquad \text{for } m \geq 1; \quad D_0(h) = -h. \tag{1.1}$$

We define an index of $h$ by

$$\operatorname{ind}_T(h) = \sum_{m=0}^{\infty} \big(\dim M^0(D_m(h)) + \dim M^-(D_m(h)) - 2N\big), \tag{1.2}$$

where $M^-(\cdot)$ and $M^0(\cdot)$ denote, respectively, the negative definite and null subspace of the self-adjoint linear operator defining it.

We have the following general result.

THEOREM 1.2. *Suppose that $H \in C^2(R^{2N}, R)$ and there exists a semipositive definite matrix $h_0 \in \mathscr{L}_s(R^{2N})$ such that $H(x) = \frac{1}{2}h_0 x \cdot x + \hat{H}(x)$ for all $x \in R^{2N}$, and $\hat{H}(x)$ satisfies $(H1)$–$(H4)$.*

*Then for every $T > 0$, the system* (HS) *possesses a nonconstant $T$-periodic solution with the minimal period not smaller than $T/(\operatorname{ind}_T(h_0) + \dim\ker h_0 + 1)$.*

As a direct consequence, we have

COROLLARY 1.3. *Suppose that H satisfies the conditions of Theorem* 1.2. *Then the following hold*:

(1) *For every* $0 < T < 2\pi/\|h_0\|$ *with* $h_0 \neq 0$, *the system* (HS) *possesses a nonconstant T-periodic solution with the minimal period not smaller than* $T/(\dim \ker h_0 + 1)$.

(2) *If* $h_0 = P^{-1} \operatorname{diag}(w_1, w_2, \ldots, w_{2N})P$ *with* $w_j \geq 0$ *for* $1 \leq j \leq 2N$ *and P being a nonsingular matrix which satisfies* $PJ = JP$, *then for any* $0 < T < 2\pi/d^{1/2}$, *where* $d = \max\{w_j w_{j+N} : 1 \leq j \leq N\}$, *the system* (HS) *possesses a periodic solution with minimal period not smaller than* $T/(\dim \ker h_0 + 1)$.

To prove these theorems, first we use the iteration formula of the Maslov-type index theory developed by Dong and Long (cf. [7]) to obtain a new relationship between the iteration number and the Maslov-type indices (Theorem 3.2). Then we use the Galerkin approximation procedure (cf. [5, 16, 27]) and the ideas used in [10] to get the conclusions.

Notice that in Theorem 1.1, Theorem 1.2, and Corollary 1.3 we give estimates on the minimal period only under the weaker condition (H4), i.e., $H''(x)$ is semipositive definite. We do not need any other strictly convex conditions on $H$. If $H$ satisfies additional conditions, we can show that the T-periodic solution has a minimal period $T$ (see Remark 4.1 and Corollary 4.2).

The paper is organized as follows. In Section 2, we compute the Maslov-type index for constant matrices and establish a relation between the index $\operatorname{ind}_T(h)$ and the Maslov-type index for the constant matrix $h$. In Section 3, we use the iteration formula of the Maslov-type index theory to estimate the iteration number and prove Theorem 3.2. This is motivated by [7]. Finally in Section 4, by using the Galerkin approximation procedure and the ideas contained in [10] and [24, 25], we prove our main results.

## 2. COMPUTATION OF THE MASLOV-TYPE INDEX

In this section, we use the notions and results in [18–20] to establish a computation formula of the Maslov-type index for any symmetric matrix $h \in \mathscr{L}_s(R^{2N})$. We obtain the relation between $\operatorname{ind}_T(h)$ and the Maslov-type index for $h \in \mathscr{L}_s(R^{2N})$ and get some useful corollaries. The idea comes from [16].

Set $Sp(2N) = \{M \in \mathscr{L}_s(R^{2N}) \mid M^T JM = J\}$. For any $T > 0$, let $S_T = R/(T\mathscr{Z})$. For $B(t) \in C(S_T, \mathscr{L}_s(R^{2N}))$, let $\gamma(t)$ be the fundamental solution of the linear Hamiltonian systems:

$$\dot{y} = JB(t)y, \quad y \in R^{2N}, \tag{2.1}$$

with $\gamma(0) = I$. Then $\gamma \in \mathscr{P}_T$, which is defined as follows:

$$\begin{aligned}\mathscr{P}_T = \big\{&\gamma \in C^1([0,T], Sp(2N)) \mid \gamma(0) = I, \dot{\gamma}(T) = \dot{\gamma}(0)\gamma(T)\\ &B(\cdot) \equiv -J\dot{\gamma}(\cdot)\gamma^{-1}(\cdot) \in C\big(S_T, \mathscr{L}_s(R^{2N})\big)\big\}.\end{aligned} \tag{2.2}$$

On the other hand, any $\gamma \in \mathscr{P}_T$ is the fundamental solution of (2.1), with $B(t)$ defined as in (2.2).

For every $\gamma \in \mathscr{P}_T$, the Maslov-type index of $\gamma$ is defined as a pair of integers,

$$(i_T, \nu_T) \equiv \big(i_T(\gamma), \nu_T(\gamma)\big) \in \mathscr{Z} \times \{0, 1, \dots, 2N\},$$

where $i_T$ is the index part and

$$\nu_T \equiv \dim\ker(\gamma(T) - I)$$

is the nullity. We also call $(i_T, \nu_T)$ the Maslov-type index of $B(t)$ if $\gamma$ is the fundamental solution of (2.1) corresponding to $B(t)$ (cf. [19, 20]).

Let $E = W^{1/2,2}(S_T, R^{2N})$. Recall that $E$ consists of those $z \in L^2(S_T, R^{2N})$ whose Fourier series

$$z(t) = a_0 + \sum_{k=1}^{\infty} \left( a_k \cos\left(\frac{2\pi}{T}kt\right) + b_k \sin\left(\frac{2\pi}{T}kt\right)\right)$$

satisfies

$$\|z\|^2 \equiv T|a_0|^2 + \frac{T}{2}\sum_{k=1}^{\infty} k\big(|a_k|^2 + |b_k|^2\big) < \infty.$$

The inner product in $E$ is given by

$$\langle z, z' \rangle = Ta_0 \cdot a_0' + \frac{T}{2}\sum_{k=1}^{\infty} k(a_k \cdot a_k' + b_k \cdot b_k'). \tag{2.3}$$

Let $\mathscr{L}_s(E)$ denote the space of the bounded self-adjoint linear operators on $E$, and let $\mathscr{L}_c(E)$ denote the space of compact operators on $E$. For $B(t) \in C(S_T, \mathscr{L}_s(R^{2N}))$, we define two operators $A, B \in \mathscr{L}_s(E)$ by extending the bilinear forms

$$\langle Ax, y\rangle = \int_0^T (-J\dot{x}, y)\, dt, \quad \langle Bx, y\rangle = \int_0^T (B(t)x, y)\, dt \tag{2.4}$$

on $E$. Then $\ker A = R^{2N}$, the Fredholm index $\operatorname{ind} A = 0$, and $B \in \mathscr{L}_c(E)$ (cf. [18]). Using the Floquet theory, we have

$$\nu_T = \dim \ker(A - B). \tag{2.5}$$

Set

$$E(m) = \left\{ z \in E \mid z(t) = a \cos\left(\frac{2\pi}{T} mt\right) + b \sin\left(\frac{2\pi}{T} mt\right), a, b \in R^{2N} \right\}$$

and

$$E_m = E(0) + E(1) + \cdots + E(m).$$

Let $P_m$ be the orthogonal projection from $E$ to $E_m$. Then $\{P_m\}$ is an approximation scheme w.r.t. $A$, i.e.,

(1) $E_0 = P_0 E = \ker A$, $E_m = P_m E$ is finite dimension for $m \geq 1$.

(2) $P_m x \to x$ as $m \to \infty$ for any $x \in E$.

(3) $P_m A = AP_m$, $\forall m \geq 0$.

For $d > 0$, we denote by $M_d^+(\cdot)$, $M_d^-(\cdot)$, and $M_d^0(\cdot)$ the eigenspace corresponding to the eigenvalue $\lambda$ belonging to $[d, +\infty)$, $(-\infty, -d]$, and $(-d, d)$ respectively, and denote by $M^+(\cdot)$, $M^-(\cdot)$, and $M^0(\cdot)$, respectively, the positive, negative definite, and null subspace of the self-adjoint linear operator defining it. For any $L \in \mathscr{L}_s(E)$ we denote $L^\# = (L \mid_{ImL})^{-1}$, and $P_m L P_m \equiv (P_m L P_m) \mid_{E_m} \colon E_m \to E_m$.

In [10], Fei and Qiu proved the following theorem.

THEOREM 2.1 [10]. *For any $B(t) \in C(S_T, \mathscr{L}_s(R^{2N}))$ with Maslov-type index $(i_T, \nu_T)$ and any constant $0 < d \leq \frac{1}{4}\|(A - B)^\#\|^{-1}$, there exists $m^* > 0$ such that for $m \geq m^*$ we have*

$$\dim M_d^+(P_m(A - B)P_m) = \tfrac{1}{2} \dim E_m - i_T - \nu_T, \tag{2.6}$$

$$\dim M_d^-(P_m(A - B)P_m) = \tfrac{1}{2} \dim E_m + i_T, \tag{2.7}$$

$$\dim M_d^0(P_m(A - B)P_m) = \nu_T, \tag{2.8}$$

*where $B$ is the operator defined by (2.4) corresponding to $B(t)$. If $B(t) = h \in \mathscr{L}_s(R^{2N})$ is a constant matrix, then (2.6)–(2.8) hold, with $M_d^*(P_m(A - B)P_m)$ being replaced by $M^*(P_m(A - B)P_m)$ for $* = +, -, 0$.*

For any $h \in \mathscr{L}_s(R^{2N})$ and any $T > 0$, let $D_m(h)$ and $\operatorname{ind}_T(h)$ be given by (1.1) and (1.2). Denote the Maslov-type index of $h$ on $[0, T]$ by $(i_T(h), \nu_T(h))$.

Our main result in this section is the following theorem.

THEOREM 2.2. *For any $T > 0$ and $h \in \mathscr{L}_s(R^{2N})$, we have*

$$i_T(h) = (\dim M^-(-h) - N) + \sum_{m=1}^{\infty} (\dim M^-(D_m(h)) - 2N),$$

$$\nu_T(h) = \dim M^0(-h) + \sum_{m=1}^{\infty} \dim M^0(D_m(h)),$$

$$\operatorname{ind}_T(h) = i_T(h_1) + \nu_T(h_1) - N.$$

*Proof.* Let $B$ be the operator defined by (2.4) corresponding to $h$. By (2.3) and (2.4), the operators $A$ and $B$ have explicit expressions:

$$Az = \sum_{m=1}^{\infty} \frac{2\pi}{T}\left(-Jb_m \cos\left(\frac{2\pi}{T}mt\right) + Ja_m \sin\left(\frac{2\pi}{T}mt\right)\right)$$

$$Bz = ha_0 + \sum_{m=1}^{\infty} m^{-1}\left(ha_m \cos\left(\frac{2\pi}{T}mt\right) + hb_m \sin\left(\frac{2\pi}{T}mt\right)\right).$$

Thus for $a, b \in R^{2N}$, we have

$$(A - B)a = -ha,$$

$$(A - B)\left(a \cos\left(\frac{2\pi}{T}mt\right) + b \sin\left(\frac{2\pi}{T}mt\right)\right)$$
$$= \left(\frac{-1}{m}ha - \frac{2\pi}{T}Jb\right)\cos\left(\frac{2\pi}{T}mt\right) + \left(\frac{2\pi}{T}Ja - \frac{1}{m}hb\right)\sin\left(\frac{2\pi}{T}mt\right). \tag{2.9}$$

By a straightforward computation (cf. [16]), we have

$$\dim M^*(P_n(A - B)P_n) = \sum_{m=0}^{n} \dim M^*(D_m(h)), \qquad * = +, -, 0. \tag{2.10}$$

By (1.1) it is easy to show that there exists $m_0 > 0$ such that for $m \geq m_0$

$$\dim M^-(D_m(h)) = 2N, \quad \dim M^0(D_m(h)) = 0. \tag{2.11}$$

Since $\ker(A - B) \subset E_{n_0}$ for some $n_0 > 0$, by Theorem 2.1 there exists $n_1 \geq n_0$ such that for $n \geq n_1$

$$\dim M^-\big(P_n(A - B)P_n\big) = \tfrac{1}{2}\dim E_n + i_T(h) \tag{2.12}$$

$$\dim M^0\big(P_n(A - B)P_n\big) = \nu_T(h). \tag{2.13}$$

Notice that

$$\dim E(0) = 2N, \quad \dim E(m) = 4N \quad \text{for } m \geq 1.$$

Combining this with (2.10)–(2.13) and (1.2) yields the conclusions. ∎

COROLLARY 2.3. *Given two matrices* $h_1, h_2 \in \mathscr{L}_s(R^{2N})$.

(i) *If* $h_1 - h_2$ *is semipositive definite, then for any* $T > 0$

$$i_T(h_1) + \nu_T(h_1) \geq i_T(h_2) + \nu_T(h_2),$$
$$i_T(h_1) - i_T(h_2) \geq \dim M^+(h_1) - \dim M^+(h_2) \geq 0.$$

(ii) *If* $h_1$ *is semipositive definite, then* $i_T(h_1) + \nu_T(h_1) \geq N$ *for any* $T > 0$. *Moreover, if* $T' \geq T > 0$, *then* $i_{T'}(h_1) + \nu_{T'}(h_1) \geq i_T(h_1) + \nu_T(h_1)$.

*Proof.* By Theorem 2.2, we have

$$i_T(h_1) - i_T(h_2) = \sum_{m=0}^{\infty}\big(\dim M^-\big(D_m(h_1)\big) - \dim M^-\big(D_m(h_2)\big)\big)$$

$$i_T(h_1) + \nu_T(h_1) - i_T(h_2) - \nu_T(h_2)$$
$$= \sum_{m=0}^{\infty}\big(\dim M^+\big(D_m(h_2)\big) - \dim M^+\big(D_m(h_1)\big)\big).$$

Notice that

$$-h_2 = -h_1 + (h_1 - h_2)$$

and

$$D_m(h_2) = D_m(h_1) + \frac{T}{2\pi}\begin{pmatrix} h_1 - h_2 & 0 \\ 0 & h_1 - h_2 \end{pmatrix}.$$

If $h_1 - h_2$ is semipositive definite, then by a straightforward computation we have the conclusion (i). The conclusion (ii) follows from (i) and the following facts. (1) For $h_2 = 0$, $i_T(h_2) + \nu_T(h_2) = N$. (2) If $T' \geq T$, then $(T'/T)h_1 - h_1$ is semipositive definite. The proof is completed. ∎

COROLLARY 2.4. *For any $T > 0$ and $h_0 \in \mathscr{L}_s(R^{2N})$, we have*

(i) *The integers $N + i_T(h_0) + \nu_T(h_0)$ and* $\dim M^+(h_0) + \dim M^0(h_0)$ *possess the same parity.*

(ii) *If $h_0$ is semipositive definite, then the integers $i_T(h_0) + \nu_T(h_0)$ and $N$ possess the same parity.*

*Proof.* For $h = \frac{T}{2\pi} h_0$, we consider the eigenvalue problem

$$\begin{pmatrix} -h & -mJ \\ mJ & -h \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{2.14}$$

By an easy computation we have

$$x_1 = \frac{-1}{m} (\lambda J + Jh) x_2$$
$$x_2 = \frac{1}{m} (\lambda J + Jh) x_1.$$

This implies

$$\begin{aligned} \left( I_{2N} + \frac{1}{m^2} (\lambda J + Jh)^2 \right) x_1 &= 0 \\ x_2 &= \frac{1}{m} (\lambda J + Jh) x_1 \end{aligned} \tag{2.15}$$

or

$$\begin{aligned} \left( I_{2N} + \frac{1}{m^2} (\lambda J + Jh)^2 \right) x_2 &= 0 \\ x_1 &= -\frac{1}{m} (\lambda J + Jh) x_2. \end{aligned} \tag{2.16}$$

If $\binom{-(1/m)(\lambda_0 J + Jh) a}{a} = y_1$ is an eigenvector corresponding to the eigenvalue $\lambda_0$, then by (2.15) we know that $y_2 = \binom{a}{(1/m)(\lambda_0 J + Jh) a}$ is also an eigenvector corresponding to $\lambda_0$, and

$$y_1^T \cdot y_2 = -\frac{1}{m} ((\lambda_0 J + Jh) a)^T \cdot a + \frac{1}{m} a^T \cdot ((\lambda_0 J + Jh) a) = 0.$$

Thus the multiplicity of each eigenvalue of (2.14) is even, and we have

$$\dim M^*(D_m(h_0)) \text{ is even for } m \geq 1 \text{ and } * = +, -, 0. \tag{2.17}$$

By Theorem 2.2 and (2.17), we get the conclusion (i). If $h_0$ is semipositive definite, then $\dim M^+(h_0) + \dim M^0(h_0) = 2N$. Thus (ii) holds. ∎

COROLLARY 2.5. 1° *For any* $h_0 \in \mathscr{L}_s(R^{2N})$, *if* $0 < T < \frac{2\pi}{\|h_0\|}$, *then*

$$i_T(h_0) = \dim M^+(h_0) - N, \quad \nu_T(h_0) = \dim M^0(h_0). \tag{2.18}$$

2° *Suppose that* $h_0 = P^{-1}\operatorname{diag}(w_1, w_2, \dots, w_{2N})P$, *where* $P$ *is a nonsingular matrix which satisfies* $PJ = JP$.

(i) *If* $w_j w_{N+j} \le 0$ *for* $1 \le j \le N$, *then* (2.18) *holds for any* $T > 0$.

(ii) *If* $d = \max\{w_j w_{N+j} : 1 \le j \le N\} > 0$, *then* (2.18) *holds for any* $0 < T < 2\pi/d^{1/2}$

*Proof.* If $0 < T < \frac{2\pi}{\|h_0\|}$, by (1.1) we have

$$\dim M^-\big(D_m(h_0)\big) = 2N, \quad \dim M^0\big(D_m(h_0)\big) = 0 \qquad \text{for } m \ge 1. \tag{2.19}$$

Then by Theorem 2.2 we have conclusion 1°.

By (2.14)–(2.16), the eigenvalues of (2.14) are the solutions of the equation

$$\left| I_{2N} + \frac{1}{m^2}(\lambda J + Jh)^2 \right| = 0.$$

Let $d_j = \frac{T}{2\pi} w_j$ for $1 \le j \le 2N$, and $h = P^{-1}\operatorname{diag}(d_1, d_2, \dots, d_{2N})P$. Since $PJ = JP$, by direct computation, the above equation implies that

$$\begin{aligned}&\big(m^2 - (\lambda + d_1)(\lambda + d_{N+1})\big)\\ &\qquad \cdot\big(m^2 - (\lambda + d_2)(\lambda + d_{N+2})\big) \cdots \big(m^2 - (\lambda + d_N)(\lambda + d_{2N})\big) = 0.\end{aligned}$$

For $1 \le j \le N$, the equation $m^2 - (\lambda + d_j)(\lambda + d_{N+j}) = 0$ has roots $\lambda_{j1}$ and $\lambda_{j2}$ satisfying

$$\lambda_{j1} \cdot \lambda_{j2} = d_j d_{N+j} - m^2. \tag{2.20}$$

If $w_j w_{N+j} \le 0$ holds for $1 \le j \le N$, then

$$\lambda_{j1} \cdot \lambda_{j2} < 0 \qquad \text{for any } T > 0 \quad \text{and} \quad 1 \le j \le N.$$

This means that (2.19) holds for any $T > 0$. Thus, conclusion (i) holds. Similarly, if $0 < T < 2\pi/d^{1/2}$, then (2.19) holds. So conclusion (ii) holds. ∎

## 3. CONTROLLING THE PERIOD VIA THE ITERATION FORMULA

In this section we consider the linear Hamiltonian systems

$$\dot{y} = JB(t)y, \quad y \in R^{2N}, \tag{3.1}$$

where $B(t) \in C(S_T, \mathscr{L}_s(R^{2N}))$. Let $\gamma\colon [0, T] \to Sp(2N)$ be the fundamental solution of (3.1) with $\gamma(0) = I$. Denote the Maslov-type index of (3.1) by $(i_T, \nu_T) = (i_T(\gamma), \nu_T(\gamma))$. For $k \in \mathscr{N}$, $B(\cdot)$ can also be viewed as defined on $S_{kT}$, so we can define the fundamental solution of (3.1) on $[0, kT]$, $\gamma_k\colon [0, kT] \to Sp(2N)$, as follows:

$$\gamma_k = \gamma(t - jT)\gamma(T)^j \qquad \text{for } jT \le t \le (j+1)T, \quad 0 \le j \le k-1. \tag{3.2}$$

We denote the corresponding Maslov-type index of (3.1) defined on $[0, kT]$ by

$$(i_{kT}, \nu_{kT}) = (i_{kT}(\gamma_k), \nu_{kT}(\gamma_k)).$$

In [7], Dong and Long proved the following theorem, which gives a relationship between $(i_{kT}, \nu_{kT})$ and $(i_T, \nu_T)$, i.e., the iteration formula of the Maslov-type index theory (cf. [7, Theorems 4.1 and 8.3]).

THEOREM 3.1 [7]. *Suppose $k \in \mathscr{N}$. Then there exist a nondegenerate perturbation path $\beta\colon [0, T] \to Sp(2N)$ and integers $\mu(\beta)$, $t_j(\beta)$ with $0 \le \mu(\beta) \le N$, $0 \le t_j(\beta) \le k-1$ for $1 \le j \le \mu(\beta)$, such that*

1° *There hold*

$$i_{kT} \ge k(i_T + \nu_T - \mu(\beta)) + \sum_{j=1}^{\mu(\beta)} 2t_j(\beta) + \mu(\beta) - \nu_{kT}, \tag{3.3}$$

$$\sum_{j=1}^{\mu(\beta)} 2t_j(\beta) \ge \nu_{kT} - \nu_T - \frac{1 + (-1)^k}{2}\omega, \tag{3.4}$$

$$\mu(\beta) \le N - \omega, \tag{3.5}$$

*where $\omega = \omega(\gamma(T))$ is a nonnegative integer given by Definition* 6.3 *in* [9].

2° *If $\mu(\beta) = N$, then the integers $\mu(\beta)$, $i_T + \nu_T$, and $i_{kT}(\beta_k)$ possess the same parity, where $\beta_k$ is defined as* (3.2) *corresponding to $\beta$.*

3° *If $\nu_{kT} = 0$, then $\nu_T = 0$, and we can choose $\beta = \gamma$ such that*

$$i_{kT} = k(i_T - \mu(\gamma)) + \sum_{j=1}^{\mu(\gamma)} 2t_j(\gamma) + \mu(\gamma).$$

*Moreover, if* $\mu(\gamma) = N$, *then the integers* $\mu(\gamma)$, $i_T$, *and* $i_{kT}$ *possess the same parity.*

Using this theorem, we can study the iteration number $k$ of the definition interval for the systems (3.1) via its Maslov-type indices. Our result reads as follows:

THEOREM 3.2. *Let* $B \in C(S_T, \mathscr{L}_s(R^{2N}))$. *If the integers* $k$, $n_k \in \mathscr{N}$ *and* $n_1 \in \mathscr{Z}$ *satisfy* $n_k \geq n_1$, $n_k \geq N$, *and the conditions*

$$i_{kT} \leq n_k + 1, \quad i_T \geq n_1, \quad i_T + \nu_T \geq N + 1, \quad \nu_T \geq 1, \tag{3.6}$$

*then* $k \leq n_k - n_1 + 1$.

*Proof.* Without loss of generality, we assume $T = 1$. Apply Theorem 3.1 to the systems (3.1); then there exists a nondegenerate path $\beta$ such that (3.3)–(3.5) hold.

We distinguish two cases.

*Case* 1. $k$ is odd. By (3.3)–(3.6) and the oddness of $k$, we obtain

$$\begin{aligned} n_k + 1 \geq i_k &\geq k\big(i_1 + \nu_1 - \mu(\beta)\big) + \mu(\beta) - \nu_1 \\ &= i_1 + (k-1)\big(i_1 + \nu_1 - \mu(\beta)\big) \\ &\geq n_1 + (k-1)(N + 1 - N) \\ &= n_1 + k - 1. \end{aligned} \tag{3.7}$$

Then $k \leq n_k - n_1 + 2$. If $k = n_k - n_1 + 2$, by (3.7) we have

$$i_1 + \nu_1 = N + 1, \quad \mu(\beta) = N. \tag{3.8}$$

This violates 2° of Theorem 3.1. Thus $k \leq n_k - n_1 + 1$.

*Case* 2. $k$ is even. By (3.3)–(3.6) and the evenness of $k$, we obtain

$$\begin{aligned} n_k + 1 \geq i_k &\geq k\big(i_1 + \nu_1 - \mu(\beta)\big) + \mu(\beta) - \nu_1 - \omega \\ &= i_1 + \big(i_1 + \nu_1 - \mu(\beta) - \omega\big) + (k-2)\big(i_1 + \nu_1 - \mu(\beta)\big) \\ &\geq n_1 + (N + 1 - N) + (k-2)(N + 1 - N) \\ &= n_1 + k - 1. \end{aligned} \tag{3.9}$$

Then $k \leq n_k - n_1 + 2$.

If $k = n_k - n_1 + 2 \geq 4$, by (3.9) we get (3.8), which violates 2° of Theorem 3.1.

If $k = n_k - n_1 + 2 = 2$, then $n_2 = n_1$. By (3.9) we have

$$n_1 + 1 \geq i_2 \geq i_1 + \nu_1 + \left(i_1 - \mu(\beta) - \omega\right)$$
$$\geq N + 1 + (n_1 - N) = n_1 + 1.$$

Combining this with $n_2 \geq N$ and (3.6) yields that

$$i_2 = N + 1, \quad i_1 = N, \quad \mu(\beta) + \omega = N \tag{3.10}$$

$$i_1 + \nu_1 = N + 1, \quad \nu_1 = 1. \tag{3.11}$$

Then using the same arguments as (9.6)–(9.10) in [7], there exists another nondegenerate perturbation path $\xi$ such that

$$\mu(\xi) = \mu(\beta) + \omega = N, \quad i_1(\xi) = N + 1, \quad \nu_1(\xi) = 0.$$

Since $\mu(\xi) = N$ and $i_1(\xi) + \nu_1(\xi) = N + 1$ have different parities, this violates 2° of Theorem 3.1.

Thus $k \leq n_k - n_1 + 1$, and the proof is complete. ∎

*Remark* 3.3. (i) Theorem 3.2 generalizes the result given by Dong and Long [7, Theorem 9.1]. In fact, if $n_k = n_1 = N$, then $k = 1$. This implies [7, Theorem 9.1]. The idea in this proof comes from [7, Theorem 9.1].

(ii) When $N = 1$, a similar result was given by Dong and Long [7, Theorem 11.1].

The following theorem was given in [17, 26]. We will use it to get the information about the Morse index.

THEOREM 3.4. *Let $E$ be a real Hilbert space with orthogonal decomposition $E = X \oplus Y$, where* $\dim X < \infty$. *Suppose $f \in C^2(E, R)$ satisfies $(PS)$ condition and the following conditions*:

(F1) *There exist $\rho, \delta > 0$ such that $f(w) \geq \delta$ for any $w \in \partial B_\rho(0) \cap Y$.*

(F2) *There exist $e \in \partial B_1(0) \cap Y$ and $r_0 > 0$ such that $f(w) \leq 0$ for any $w \in \partial Q$, where $Q = (B_{r_0}(0) \cap X) \oplus re\colon 0 \leq r \leq r_0$, $B_r(0) = w \in E\colon \|w\| \leq r$.*

*Then*

(1) *$f$ possesses a critical value $c \geq \delta$, which is given by*

$$c = \inf_{h \in \Gamma} \sup_{w \in Q} f(h(w)),$$

*where $\Gamma = \{h \in C(Q, E) : h = id$ on $\partial Q\}$.*

(2) *There exists $w_0 \in \mathscr{K}_c \equiv w \in E\colon f'(w) = 0$, $f(w) = c$ such that the Morse index $m^-(w_0)$ of $f$ at $w_0$ satisfies*

$$m^-(w_0) \leq \dim X + 1.$$

## 4. MINIMAL PERIOD ESTIMATE OF HAMILTONIAN SYSTEMS

In this section, we study the minimal period problem of the Hamiltonian systems

$$\dot{x} = JH'(x) \qquad \forall x \in R^{2N}. \tag{HS}$$

For $T > 0$, as in Section 2, let $E = W^{1/2,2}(S_T, R^{2N})$, and let $\{P_m\}$ be the approximation scheme w.r.t. $A$, which is defined by (2.4). If $x$ is a T-periodic solution of (HS), then the Maslov-type index of the solution $x$ is defined to be the Maslov-type index of $B(t) = H''(x(t))$ and is denoted by $(i_T(x), \nu_T(x))$, just as in [7, 19].

For $z \in E$, we define

$$f(z) = \tfrac{1}{2}\langle Az, z\rangle + \int_0^T H(z)\, dt. \tag{4.1}$$

It is well known that $f \in C^2(E, R)$ whenever $H \in C^2(R^{2N}, R)$ and satisfies

(H5) $|H''(x)| \leq a|x|^s + b$, for $s \in (1, \infty)$, $a, b > 0$ and all $x \in R^{2N}$.

Looking for solutions of (HS) is equivalent to looking for critical points of $f$ (cf. [5, 27]). We first prove Theorem 1.2.

*Proof of Theorem* 1.2. We carry out the proof in several steps.

Step 1. (i) Since the growth condition (H5) has not been assumed for $\hat{H}$, we truncate the function $\hat{H}$ suitably to get a function $\hat{H}_K$ which satisfies (H5), just as in [24, 25]. We define $f_K$ by (4.1) with $H(z) = \frac{1}{2}h_0 z \cdot z + \hat{H}_K(z)$, then $f_K \in C^2(E, R)$ (cf. [10, Theorem 3.1 (Step 1)]).

(ii) For $m > 0$, write $f_{Km} = f_K|_{E_m}$. One can show that $f_{Km}$ satisfies the hypotheses of Theorem 3.4 (cf. [24, 35]). Then, by using the Galerkin approximation procedure and Theorem 2.1, one can show that there is a nonconstant classical solution $x_0$ of (HS) which satisfies

$$i_T(x_0) \leq i_T(h_0) + \nu_T(h_0) + 1 \tag{4.2}$$

(cf. [24, 25], [10, Theorem 3.1 (Step 2 through Step 4)]).

Step 2. If $x_0$ has minimal period $\tau = T/k$ with some positive integer $k$, we want to prove that

$$i_\tau(x_0) \geq N - \dim\ker h_0 \tag{4.3}$$

$$i_\tau(x_0) + \nu_\tau(x_0) \geq N + 1. \tag{4.4}$$

Let $E_\tau = W^{1/2,2}(S_\tau, R^{2N})$, let $A_\tau$ be the operator defined by (2.4) on $E_\tau$, let $\{P_{\tau m}\}$ be the usual approximation scheme w.r.t. $A_\tau$ (as in Section 2), and let $B_\tau$ be the operator defined by (2.4) on $[0, \tau]$ corresponding to $B(t) = H''(x_0(t))$.

For $z \in E_\tau$, set

$$f_\tau(z) = \tfrac{1}{2}\langle (A_\tau - B_\tau)z, z\rangle = \tfrac{1}{2}\langle A_\tau, z, z\rangle - \tfrac{1}{2}\int_0^\tau H''\big(x_0(t)\big)z \cdot z\,dt$$

and

$$f_{\tau m}(w) = f_\tau(w) \qquad \forall w \in E_{\tau m} = P_{\tau m}E_\tau.$$

Notice that $\ker h_0 \subset R^{2N} = M^0(A_\tau)$. Let

$$X = \left\{z \in \ker h_0 : \hat{H}''\big(x_0(t)\big)z = 0, \forall t \in S_\tau\right\},$$

and let $Y$ be the orthogonal complement of $X$ in $R^{2N}$, i.e., $R^{2N} = X \oplus Y$. Since $H''(x_0(t)) = h_0 + \hat{H}''(x_0(t))$, by (H4), it is easy to show that there exists $\lambda_0 > 0$ such that

$$\int_0^T H''\big(x_0(t)\big)z_0 \cdot z_0\,dt \geq \lambda_0\|z_0\|^2, \qquad \forall z_0 \in Y.$$

Thus for any $z = z_- + z_0 \in P_{\tau m}M^-(A_\tau) \oplus Y$ with $\|z\| = 1$, we have that

$$\begin{aligned} f_{\tau m}(z) &= \frac{1}{2}\langle (A_\tau - B_\tau)z, z\rangle = \frac{1}{2}\langle A_\tau z_-, z_-\rangle - \frac{1}{2}\int_0^\tau H''\big(x_0(t)\big)z \cdot z\,dt \\ &\leq -\frac{1}{2}\|A_\tau^{\#}\|^{-1}\|z_-\|^2 - \frac{1}{2k}\int_0^T H''\big(x_0(t)\big)z_0 \cdot z_0\,dt \\ &\quad - \int_0^\tau H''\big(x_0(t)\big)z_- \cdot z_0\,dt \\ &\leq -\alpha\|z_-\|^2 - \gamma\|z_0\|^2 + \beta\|z_-\|\,\|z_0\|, \end{aligned} \tag{4.5}$$

where $\alpha = \frac{1}{2}\|A_\tau^{\#}\|^{-1}$, $\beta = \max_{t \in [0,\tau]}|H''(x_0(t))|$, and $\gamma = \lambda_0(2k)^{-1} > 0$ are independent of $m$. It is easy to show that there exists $0 < c_0 < 1$ such that if $\|z_0\| \geq c_0$,

$$f_{\tau m}(z) \leq -\alpha\|z_-\|^2 - \gamma\|z_0\|^2\Big(1 - \beta\gamma^{-1}\big(\|z_0\|^{-2} - 1\big)^{1/2}\Big) \leq \frac{-\gamma}{2}c_0^2.$$

If $\|z_0\| < c_0$, then $\|z_-\|^2 = 1 - \|z_0\|^2 \geq 1 - c_0^2$. By (H4) we have

$$f_{\tau m}(z) \leq -\alpha\|z_-\|^2 \leq -\alpha\left(1 - c_0^2\right).$$

Hence

$$f_{\tau m}(z) \leq -c_1\|z\|^2, \qquad \forall z \in P_{\tau m}M^-(A_\tau) \oplus Y, \tag{4.6}$$

where $c_1 = \min\{\frac{\gamma}{2}c_0^2, \alpha(1 - c_0^2)\}$ is independent of $m$. Let

$$d = \min\left\{\frac{1}{4}\|(A_\tau - B_\tau)^{\#}\|^{-1}, \frac{c_1}{2}\right\}.$$

By (4.6) and Theorem 2.1, for $m$ large enough, we have

$$\begin{aligned}\tfrac{1}{2}\dim E_{\tau m} + i_\tau(x_0) &= \dim M_d^-\left(P_{\tau m}(A_\tau - B_\tau)P_{\tau m}\right)\\ &\geq \dim\left(P_{\tau m}M^-(A_\tau) \oplus Y\right)\\ &= \left(\tfrac{1}{2}\dim E_{\tau m} - N\right) + (2N - \dim X).\end{aligned}$$

This implies that

$$i_\tau(x_0) \geq N - \dim X. \tag{4.7}$$

Notice that $\dim X \leq \dim\ker h_0$, and we get (4.3).

For any $z_0 \in X$, by the definition of $X$, $z_0$ is a solution of the linear Hamiltonian system

$$\dot{z} = JH''\left(x_0(t)\right)z. \tag{4.8}$$

Since $x_0(t)$ is a nonconstant $\tau$-periodic solution of (HS), we know that $\dot{x}_0(t)$ is a nonconstant $\tau$-periodic solution of (4.8). Hence we have that

$$\nu_\tau(x_0) \geq \dim X + 1. \tag{4.9}$$

By (4.7) and (4.9), we get (4.4).

Step 3 By (4.9) we have that $\nu_\tau(x_0) \geq 1$. If $\tau = T/k$ is the minimal period of $x_0$, then by (4.2)–(4.4), Theorem 3.2, and Theorem 2.2 we have that

$$\begin{aligned}k &\leq i_T(h_0) + \nu_T(h_0) - N + \dim\ker h_0 + 1\\ &\leq \operatorname{ind}_T(h_0) + \dim\ker h_0 + 1.\end{aligned}$$

This means that

$$\tau \geq T/\left(\operatorname{ind}_T(h_0) + \dim\ker h_0 + 1\right).$$

∎

*Proof of Theorem* 1.1. We consider Theorem 1.1 as a special case of Theorem 1.2 with $h_0 = 0$.

If $h_0 = 0$, by Theorem 2.2 we have that $i_T(h_0) + \nu_T(h_0) = N$, $\dim\ker h_0 = 2N$. By (4.9) we have

$$\dim X \leq \nu_\tau(x_0) - 1 \leq 2N - 1.$$

Combining this with (4.7) yields that $i_\tau(x_0) \geq 1 - N$. Now by Theorem 3.2 we have

$$k \leq N - (1 - N) + 1 = 2N,$$

i.e., the minimal period $\tau$ of $x_0$ satisfies $\tau \geq T/(2N)$. ∎

*Proof of Corollary* 1.3. By Theorem 2.2 and Corollary 2.5, under the conditions of (1) and (2), we always have that $\operatorname{ind}_T(h_0) = 0$. The conclusions come directly from Theorem 1.2. ∎

*Remark* 4.1. In the proof of Theorem 1.2, by (4.2), (4.7), and Theorem 3.2, we get

$$k \leq \operatorname{ind}_T(h_0) + \dim X + 1,$$

where $X = \{z \in \ker h_0 : \hat{H}''(x_0(t))z = 0, \forall t \in S_\tau\}$. Therefore if $\operatorname{ind}_T(h_0) = \dim X = 0$, the corresponding solution $x_0(t)$ has minimal period $T$. This suggests the following corollary.

Set

$$\mathscr{H}_N = \left\{h \in \mathscr{L}_s(R^{2N}) \mid \operatorname{ind}_T(h) = 0 \text{ holds for any } T > 0\right\}. \quad (4.10)$$

Obviously, $0 \in \mathscr{H}_N$. There are many nonzero matrices in $\mathscr{H}_N$. For example, if $h_0 = P^{-1}\operatorname{diag}(w_1, w_2, \ldots, w_{2N})P$, where $P$ is a nonsingular matrix which satisfies $PJ = JP$, and $w_j \geq 0$ for $1 \leq j \leq 2N$ and $w_j w_{N+j} = 0$ for $1 \leq j \leq N$, then $h_0 \in \mathscr{H}_N$.

COROLLARY 4.2. *Suppose that $H$ satisfies the conditions of Theorem* 1.2 *and*

(H6) *The set $D = \{x \in R^{2N} : H'(x) \neq 0,\ \hat{H}''(x)z_0 = 0$ for some nonzero $z_0 \in \ker h_0\}$ is hereditarily disconnected, i.e., every connected component of $D$ contains only one point.*

*Then for every $T > 0$, the system* (HS) *possesses a nonconstant $T$-periodic solution with the minimal period not smaller than $T/(\operatorname{ind}_T(h_0) + 1)$. Moreover, the following hold*:

(i) *For every $0 < T < 2\pi/\|h_0\|$ with $h_0 \neq 0$, the system* (HS) *possesses a nonconstant $T$-periodic solution with the minimal period $T$.*

(ii) *If* $h_0 \in \mathscr{H}_N$, *for every* $T > 0$, *the system* (HS) *possesses a nonconstant T-periodic solution with the minimal period* $T$.

(iii) *If* $h_0 = P^{-1}\,\mathrm{diag}(w_1, w_2, \ldots, w_{2N})P$, *where* $w_j \geq 0$ *for* $1 \leq j \leq 2N$ *and* $P$ *is a nonsingular matrix which satisfies* $PJ = JP$, *then for any* $0 < T < 2\pi/d^{1/2}$, *where* $d = \max\{w_j w_{j+N} : 1 \leq j \leq N\}$, *the system* (HS) *possesses a periodic solution with minimal period* $T$.

*Proof.* By Remark 4.1, Theorem 2.2, and Corollary 2.5, we only need to show that the condition (H6) implies that $\dim X = 0$.

In fact, suppose that $\dim X \neq 0$. Then there exist nonzero $z_0 \in X$. Since $x_0(t)$ is a nonconstant classical solution of (HS), there exist $t_0$ and $t_1$ such that $[t_0, t_1] \subset [0, T]$ and $\dot{x}_0(t) \neq 0$ for all $t \in [t_0, t_1]$. This implies that $H'(x_0(t)) \neq 0$ for all $t \in [t_0, t_1]$. By the definition of $X$ and $D$ (see (H6)), we know that $x_0(t) \in D$ for all $t \in [t_0, t_1]$. This implies that $D$ is not hereditarily disconnected. Therefore we must have $\dim X = 0$. ∎

## REFERENCES


1. H. Amann and E. Zehnder, Nontrivial solutions for a class of nonresonance problem and applications to nonlinear differential equations, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (*4*) **7** (1980), 539–603.
2. A. Ambrosetti and V. Coti Zelati, Solutions with minimal period for Hamiltonian systems in a potential well, *Ann. Inst. H. Poincare Anal. Non Lineaire* **4** (1987), 275–296.
3. A. Ambrosetti and G. Mancini, Solutions of minimal period for a class of convex Hamiltonian systems, *Math. Ann.* **255** (1981), 405–421.
4. C. Conley and E. Zehnder, Morse type index theory for flows and periodic solutions for Hamiltonian equations, *Comm. Pure Appl. Math.* **37** (1984), 207–253.
5. K. C. Chang, Infinite dimensional Morse theory and multiple solution problems, *in* "Progress in Nonlinear Differential Equations and their Applications," Springer-Verlag, Berlin, 1993.
6. F. Clarke and I. Ekeland, Hamiltonian trajectories having prescribed minimal period, *Comm. Pure Appl. Math.* **33** (1980), 103–116.
7. D. Dong and Y. Long, The iteration formula of the Maslov-type index theory with applications to nonlinear Hamiltonian systems, *Trans. Amer. Math. Soc.* **349** (1997), 2619–2661.
8. I. Ekeland, "Convexity Method in Hamiltonian Mechanics," Springer-Verlag, Berlin, 1990.
9. I. Ekeland and H. Hofer, Periodic solutions with prescribed period for convex autonomous Hamiltonian systems, *Invent. Math.* **81** (1985), 155–188.
10. G. Fei and Q. Qiu, Minimal periodic solutions of nonlinear Hamiltonian systems, *Nonlinear Anal.* **27** (1996), 821–839.
11. N. Ghoussoub, Location, multiplicity and Morse indices of min-max critical points, *J. Reine Angrew. Math.* **417** (1991), 27–76.
12. M. Giradi and M. Matzeu, Some results on solution of minimal period to superquadratic Hamiltonian equations, *Nonlinear Anal.* **7** (1983), 475–482.
13. M. Giradi and M. Matzeu, Solution of minimal period for a class of nonconvex Hamiltonian systems and applications to the fixed energy probem, *Nonlinear Anal.* **10** (1986), 371–382.

14. M. Giradi and M. Matzeu, Periodic solutions of convex autonomous Hamiltonian systems with a quadratic growth at the origin and superquadratic at infinity, *Ann. Mat. Pura Appl.* (*4*) **147** (1987), 21–72.
15. M. Giradi and M. Matzeu, Essential critical points of linking type and solution of minimal period to superquadratic Hamiltonian systems, preprint.
16. S. Li and J. Q. Liu, Morse theory and asymptotically linear Hamiltonian systems, *J. Differential Equations* **78** (1989), 53–73.
17. A. Lazer and S. Solimini, Nontrivial solution of operator equations and Morse indices of critical points of min-max type, *Nonlinear Anal.* **12** (1988), 761–775.
18. Y. Long and E. Zehnder, Morse theory for forced oscillations of asymptotically linear Hamiltonian systems, *in* "Stochastic Processes, Physics and Geometry, Proceedings of the Conference in Asconal/Locarno, Switzerland" (S. Albeverio et al., Eds.), pp. 528–563, World Scientific, Singapore, 1990.
19. Y. Long, Maslov-type index, degenerate critical points and asymptotically linear Hamiltonian systems, *Sci. China Ser. A* **33** (1990), 1409–1419.
20. Y. Long, "The Index Theory of Hamiltonian Systems with Applications," Science Press, Beijing, 1993 (in Chinese).
21. Y. Long, The minimal period problem of periodic solutions for autonomous superquadratic second order Hamiltonian systems, *J. Differential Equations* **111** (1994), 147–174.
22. Y. Long, The minimal period problem of classical Hamiltonian systems with even potentials, *Ann. Inst. H. Poincaré Anal. Non Lineaire* **10**, No. 6 (1993), 605–626.
23. J. Mawhin and M. Willem, "Critical Point Theory and Hamiltonian Systems," Springer-Verlag, New York/Berlin, 1989.
24. P. H. Rabinowitz, Periodic solutions of Hamiltonian systems, *Comm. Pure Appl. Math.* **31** (1978), 157–184.
25. P. H. Rabinowitz, Minimax methods in critical point theory with applications to differential equations, *CBMS Regional Conf. Ser. in Math.* No. 65 (1986).
26. S. Solimini, Morse index estimates in min-max theorems, *Manuscripta Math.* **63** (1989), 421–453.
27. A. Szulkin, Cohomology and Morse theory for strongly indefinite functionals, *Math. Z.* **209** (1992), 375–418.